Midterm 2

Due Wednesday, April 20$^{\text{th}}$

Please do at least 5 problems in total, including one from each section. If you do more than 5, please indicate the 5 you would like us to grade first; if one of those 5 is badly wrong, we will replace it with another from the same section.

1 Models

Recall this definition from the practice exam:

Definition 1. If $\mathcal{A}$ is a σ-structure, we say $B \subseteq A$ is a substructure if:

- For each constant $c \in C_\sigma$, $c_{\mathcal{A}} \in B$
- For each n-ary function $f \in F_\sigma$ and $b_1, \ldots, b_n \in B$, $f_{\mathcal{A}}(b_1, \ldots, b_n) \in B$

In this case, we make B is a σ-structure $\mathcal{B}$, interpreting functions and relations like in $\mathcal{A}$. Explicitly:

- For $c \in C_\sigma$, $c_{\mathcal{B}} = c_{\mathcal{A}}$
- For $f \in F_\sigma$, $f_{\mathcal{B}}(b_1, \ldots, b_n) = f_{\mathcal{A}}(b_1, \ldots, b_n)$
- For $R \in R_\sigma$, $R_{\mathcal{B}} = \{(b_1, \ldots, b_n) \in B^n | (b_1, \ldots, b_n) \in R_{\mathcal{A}}\}$

1. Let $\mathcal{B} \subseteq \mathcal{A}$ be a substructure, and $s \colon V \to B$ be a $\mathcal{B}$-valued assignment function. Since $B \subseteq A$, we can view s as an $\mathcal{A}$-valued assignment function as well, which we will call s' (formally, $s' = \iota \circ s$, where ι is the inclusion function from B to A).
   Show that for any σ-term τ, $\overline{s}(\tau) = \overline{s'}(\tau)$

2. Suppose σ is algebraic, T is an equational theory, and $\mathcal{A} \models T$. Let $\mathcal{B}$ be a substructure of $\mathcal{A}$. Show that $\mathcal{B} \models T$ (you may assume problem 1)

   Recall that a signature is algebraic if it contains no relations, and a theory is equational if every φ in T looks like:

   $$\forall x_{i_1} \forall x_{i_2} \ldots \forall x_{i_n} \ (\tau_1 = \tau_2)$$

   for some σ-terms τ_1 and τ_2.

2 Deductions

3. Let σ be the signature with one unary relation P. Give a formal deduction to show $\forall x Px \vdash \neg\forall y \neg Py$ (do not quote meta-theorems)

   Remark: To avoid a problem some of you pointed out previously, I recall that our deductive system always assumes structures are non-empty to avoid pathologies.

4. Let $\sigma = (a, b, \cdot)$, where a and b are constants, and $\cdot$ is a binary function. Let T be the theory asserting a and b are left and right identities for $\cdot$:

   1. $\forall x(x \cdot a = x)$
   2. $\forall x(a \cdot x = x)$
   3. $\forall x(x \cdot b = x)$
   4. $\forall x(b \cdot x = x)$

   Show that $T \vdash a = b$. This one is a bit longer, to so you can quote meta-theorems or results from class if you'd like.

3 Soundness and Completeness

To avoid repitition, let $\sigma_{part} = (P, Q)$, two unary relations, and T_{part} is the theory of a set partitioned into disjoint sets P and Q; that is,

$$T_{part} = \{\forall x(Px \vee Qx), \quad \forall x \neg(Px \wedge Qx)\}$$

5. Consider the σ_{part} theory T defined as follows:

   Let α_n be the statement $\exists x_1 \exists x_2 \ldots \exists x_n (Px_1 \wedge \cdots \wedge Px_n \wedge [\text{no two } x_i\text{'s are equal}])$
   Let β_n be the statement $\exists x_1 \exists x_2 \ldots \exists x_n (Qx_1 \wedge \cdots \wedge Qx_n \wedge [\text{no two } x_i\text{'s are equal}])$

   Finally, let $T = T_{part} \cup \left(\bigcup_{n \in \mathbb{N}} \{\alpha_n, \beta_n\} \right)$

   (a) Show that T is consistent

   (b) Show that T is complete

(c) Let ω_1 be the smallest uncountable cardinal (that is, for any infinite $A \subseteq \omega_1$, $|A| = |\mathbb{N}|$ or $|A| = |\omega_1|$). Up to isomorphism, how many models does T have of cardinality ω_1?

6. Let $\sigma = (P, Q, \leq)$. Consider the following theory:

$$\begin{aligned} T = \; & T_{part} & \cup \\ & T_{dlo} & \cup \\ & \{\forall x \exists y \exists z (Py \wedge Pz \wedge y < x \wedge x < z)\} & \cup \\ & \{\forall x \exists y \exists z (Qy \wedge Qz \wedge y < x \wedge x < z)\} & \end{aligned}$$

(where $x < y$ is shorthand for $x \leq y \wedge \neg(x = y)$)

(a) Show T is consistent

(b) Show T is not complete
Hint: Show the statement
$\forall x \forall y((x < y \wedge Px \wedge Py) \to \exists z(Pz \wedge x < z \wedge z < y))$
can't be proved or disproved

(c) (Hard and optional) Find some axioms you can add to T to make T complete and consistent

7. The goal of this problem is to show connected graphs are not axiomatizable (in the signature with one binary edge relation). I don't care how you do this, but I'll suggest two methods. Use whichever one makes more sense to you.

(a) Let G be the graph with verticies $\mathbb{N}$ and edges $\{n, n+1\}$ (so, an infinite stick in one direction)

i. Let T be the complete theory of G (that is, the set of true statements in G). Show T has a model other than G (use compactness to add a new element)

ii. Show that any model of T other than G is disconnected (show that any new elements don't have a path connecting them to 0)

iii. If Σ axiomatized connected groups, show $\Sigma \subseteq T$, and therefore Σ has a disconnected model

(b) Let σ^+ be the signature (E, a, b), where a and b are constants

i. Give a σ^+ theory T axiomatizing "graphs with no path connecting a and b"

ii. Show that every finite subset of T is satisfiable in a connected graph.

iii. If Σ axiomatized connected graphs, show that every finite subset of $\Sigma \cup T$ has a model. Conclude by compactness that $\Sigma \cup T$ has a model, which is both connected and disconnected.