Contents

Start by doing the following exercises:
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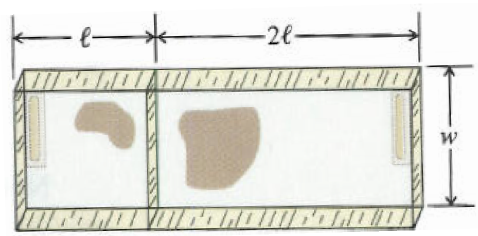
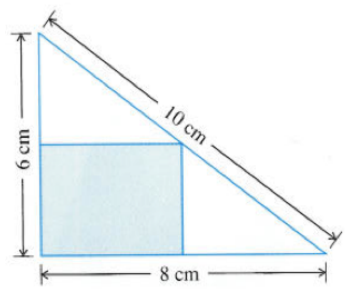
4 Applications of the Derivative

4.4 Optimization Problems



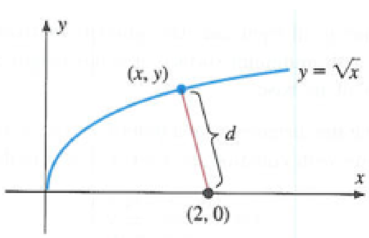
1. A farmer has 120 meters of fencing with which he plans to make a rectangular pig pen. The pen will have one internal fence that runs parallel to the end fences and divides the pen into two sections. Calculate the dimensions that produce the pen of maximum area assuming that the length of the larger section is to be twice the length of the smaller section (see the figure). Justify the fact that your answer corresponds to the pen of maximum area.



2. A rectangle is inscribed in a right triangle with sides of length 6 cm, 8 cm, and 10 cm, respectively. Calculate the dimensions of the rectangle of maximum area if two sides of the rectangle lie along two sides of the triangle (see the figure). Justify the fact that your answer corresponds to the rectangle of maximum area.

3. An orchard presently has 25 trees per acre. The average yield has been calculated to be 495 apples per tree. For each additional tree planted per acre, it has been predicted that the yield will decrease by 15 apples per tree. Should additional trees be planted to increase the yield? If so, how many should be planted to maximize the yield? Include a justification for your answer.

4. Calculate the point on the graph of $y = \sqrt{x}$ that is nearest to the point $(2, 0)$. Include a justification that this point corresponds to a minimum.



4.5 Linear Approximation

1. Consider the statement: "If the graph of $y = g(x)$ is a line tangent to the graph $y = f(x)$ at the point $x = a$, then g gives a good approximation for f near $x = a$." Do you agree with this statement? Justify your response.

2. Let $f(x) = x^2$ and $g(x) = 2x - 1$.

   (a) Confirm that the graph of g is tangent to the graph of f at the point $(1, 1)$.

   (b) Sketch the graphs of f and g over the interval $-1 \leq x \leq 3$.

   (c) Note that $g(x)$ can also be expressed in terms of the Δx notation as $g(x) = 1 + 2\,\Delta x$ where $\Delta x = x - 1$ in this case.

   (d) Use the equation $g(x)$ of the tangent line or equivalently the Δx notation to compute an approximate value for $f(1.06)$. Use a calculator or even multiplication by hand (!) to find the exact value of $f(1.06)$. How close is your approximation to the exact value?

   (e) What is the maximum value of $|f(x) - g(x)|$ on this interval? What is the significance of this value in terms of the linear approximation?

3. (a) Use the equation of the tangent line to $f(x) = \sqrt{x}$ at $a = 4$ to approximate the value of $\sqrt{4.1}$.

   (b) Use a calculator to calculate the error in this approximation (to the nearest ten thousandth).

4. Use linear approximation to estimate the value of $e^{0.03}$. Clearly show all work.

5. (a) Find the linearization at $a = 0$ of each of the following functions:

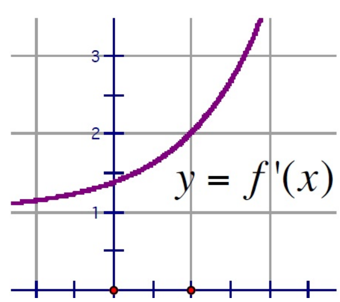
$$f(x) = (x-1)^2 \qquad g(x) = e^{-2x} \qquad h(x) = 1 + \ln(1 - 2x)$$

   What do you notice? Explain your observation.

   (b) Graph each function and its linear approximation. (You may use a calculator or a computer for this.)

   (c) For which function is the linear approximation best/worst? Does the answer depend on the chosen interval? Explain.

6. Suppose we know $f(1) = 5$ and the graph of $y = f'(x)$ is as shown on the right.



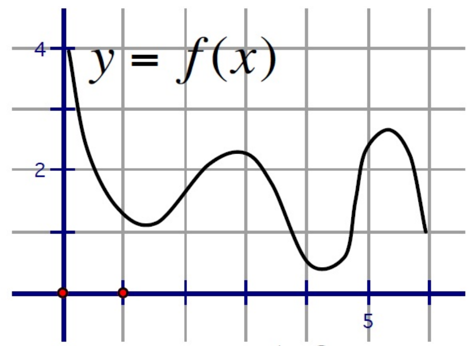
   (a) Approximate the values $f(0.9)$ and $f(1.1)$.

   (b) Can you tell if these estimates are too large or too small? Explain.

4.6 The Mean Value Theorem

1. (a) Before applying the Mean Value Theorem, what hypotheses must be checked?
   (b) If the hypotheses of the Mean Value Theorem are satisfied, what conclusion can you make?

2. Use the graph of $y = f(x)$ shown on the right to estimate the values of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 6]$.



3. True or False? For the function $f(x) = x^2$ on the interval $[-1, 2]$, there is a number c in the open interval $(-1, 2)$ such that
$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$
If true, compute all such numbers c. If false, explain.

4. True or False? For the function $f(x) = |x|$ on the interval $[1, 2]$, there is a number c in $(-1, 2)$ such that
$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}.$$
If true, compute all such numbers c. If false, explain.

5. A pumpkin is dropped from a height of 64 feet. Its height s, in feet, is given by $s(t) = -16t^2 + 64$ where time t is given in seconds.
   (a) Determine the interval on which the function is valid. In other words, when does the pumpkin hit the ground?
   (b) Apply the Mean Value Theorem to establish that there is a time in this interval when the pumpkin is traveling downwards at a speed of exactly 32 ft/sec.

Challenge Problems

1. (a) Explain why the Mean Value Theorem does not apply to $f(x) = \tan x$ on $[0, \pi]$.
   (b) Explain why the Mean Value Theorem does not apply to $g(x) = x^{2/3}$ on $[-1, 1]$.

2. Prove/Explain: If f is differentiable on $(-\infty, \infty)$, then between any two x-intercepts of f there exists at least one x-intercept of f'.

3. Two racers start a race at the same moment and finish in a tie. Which statement must be true?
   (a) At some point during the race, the two racers were not tied.
   (b) The racers' speeds at the end of the race must have been exactly the same.
   (c) The racers must have had the exact same speed at the exact same time during the race.
   (d) There exist times t_1 and t_2 such that the first racer's speed at t_1 was equal to the second racer's speed at t_2, but $t_1 \neq t_2$.