Contents

Start by doing the following exercises:
§ 4.7: #1, 2, 3, 5, 6, 9, 10, 13 § 4.9: #1, 2, 4, 5, 7, 10

4 Applications of the Derivative

4.7 L'Hôpital's Rule

1. (a) Before you apply L'Hôpital's Rule, what hypotheses must you check?
   (b) If the hypotheses of L'Hôpital's Rule are met, what conclusion(s) can you make?

2. (a) Simplify the fraction $(x^2 - 9)/(x - 3)$. Then use the result to calculate
   $$\lim_{x\to 3} \frac{x^2 - 9}{x - 3}.$$
   Note that this method of calculating the limit is the one we discussed at the start of the semester.
   (b) As an alternative to the computation you made in part (a), apply L'Hôpital's Rule to evaluate the limit.
   (c) Calculate the limit
   $$\lim_{x\to 3} \frac{x^2 - 9}{x + 3}.$$
   What theorem(s) justify your calculation?
   (d) Explain why L'Hôpital's Rule does *not* help us evaluate the limit in part (c).

3. Calculate the following limits.
   (a) $\lim_{x\to 1} \frac{1 - x}{e^x - e}$ (b) $\lim_{x\to 0} \frac{\sin x^2}{x}$ (c) $\lim_{x\to\infty} \frac{e^{6x}}{x^2}$ (d) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

4. What happens if you try to apply L'Hôpital's Rule to the limit
   $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}\,?$$
   Use another method to evaluate this limit.

5. Explain how to convert a limit of the form $\infty \cdot 0$ to a limit of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

6. Consider the limit $\lim_{x\to\infty} xe^{-x}$.

   (a) Show that this limit can be rewritten so that we obtain an indeterminate form of type $\frac{\infty}{\infty}$.

   (b) Use L'Hôpital's Rule to evaluate the limit.

7. (a) State the indeterminate form of $\lim_{x\to 0^+} x^2 \ln x$.

   (b) Rewrite the expression and apply L'Hôpital's Rule to evaluate the limit.

8. In parts (a)–(c), evaluate the limit $\lim_{x\to\infty} f(x)g(x)$.

   (a) Let $f(x) = x^{-1}$ and $g(x) = x$.

   (b) Let $f(x) = x^{-1}$ and $g(x) = x^2$.

   (c) Let $f(x) = x^{-2}$ and $g(x) = x$.

   (d) Use the results of parts (a)–(c) to explain why $\infty \cdot 0$ is an indeterminate form.

9. (a) Explain the procedures we use to evaluate limits of the form 0^0, 1^∞, or ∞^0.

   (b) State the indeterminate form of the limit $\lim_{x\to 0^+} (1+x)^{1/x}$. Then evaluate the limit.

   (c) State the indeterminate form of the limit $\lim_{x\to\infty} (1+x)^{e^{-x}}$. Then evaluate the limit.

10. In parts (a)–(c), evaluate the limits.

    (a) $\lim_{x\to\infty} \left(x^2 - x\right)$ (b) $\lim_{x\to\infty} \left(xe^{1/x} - x\right)$ (c) $\lim_{x\to\pi/2^-} (\sec x - \tan x)$

    (d) Use the results of parts (a)–(c) to explain why $\infty - \infty$ is an indeterminate form.

11. (a) What does it mean to say that $f(x)$ grows faster than $g(x)$ as $x \to \infty$?

    (b) Use a calculator or computer to graph $f(x) = \sqrt{x}$ and $g(x) = 1 + \ln x$ on the same set of axes.

    (c) Based on the graph, which function appears to grow faster as $x \to \infty$?

    (d) Use limits to compare the growth rates of f and g as $x \to \infty$.

12. (a) Use a calculator or computer to graph $f(x) = \dfrac{e^x}{2x}$ and $g(x) = x^2$ on the same set of axes.

    (b) Based on the graph, which function appears to grow faster as $x \to \infty$?

    (c) Use limits to compare the growth rates of f and g as $x \to \infty$.

13. Evaluate the following limits. Clearly state the indeterminate forms involved and the results you use. (Hint: In part (c), use the change of variables $x = 1/t$.)

    (a) $\lim_{x\to 0^+} \sin x \ln x$ (b) $\lim_{x\to 0^+} x \ln\left(\dfrac{x}{x+1}\right)$ (c) $\lim_{x\to\infty} x \ln\left(\dfrac{x}{x+1}\right)$

14. The functions f and g are differentiable on $(-\infty, \infty)$. The equation of the tangent to the graph of $y = f(x)$ at $x = 2$ is $y = -x + 2$. The equation of the tangent to the graph of $y = g(x)$ at $x = 2$ is $y = 3x - 6$. Evaluate the limit
$$\lim_{x\to 2} \frac{f(x)}{g(x)}.$$
Write a one- or two-sentence justification of your reasoning.

Challenge Problems

1. Consider the limit $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{x}$.
   (a) Show that L'Hôpital's Rule does not help us to determine this limit.
   (b) Use methods from earlier this semester to evaluate the limit.
   (c) Why doesn't this example contradict L'Hôpital's Rule? In other words, what phrase in the statement of the theorem is not satisfied?

2. Show that an immediate application of L'Hôpital's Rule may not help us evaluate
$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$
Nevertheless, show that we can rewrite the expression and then apply L'Hôpital's Rule to find the limit.

3. The compound interest formula
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
is used to calculate the account balance A in a bank account after t years, where P is the initial investment, r is the annual interest rate, and n is the number of times per year that interest is compounded.
   (a) Justify this formula.
   (b) Calculate the limit of this formula as $n \to \infty$. Show that this limit is the continuous interest formula $A = Pe^{rt}$.

4. Suppose f' is continuous, $f(2) = 0$, and $f'(2) = 5$. Evaluate $\lim\limits_{x\to 0} \dfrac{f(3x+2) + f(5x+2)}{x}$.

5. Consider the function
$$f(x) = \begin{cases} |x|^x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$
(You may find it informative to use a computer or calculator to examine the graph of f near $x = 0$.)
   (a) Show that f is continuous at $x = 0$.
   (b) Show that f is not differentiable at $x = 0$.

4.9 Antiderivatives

1. Fill in the blanks with the appropriate word or phrase:
   (a) If $F'(x) = f(x)$, then f is the ______________ of F and F is an ______________ of f.
   (b) If f is an antiderivative of g and g is an antiderivative of h, then h is the ______________ of f.

2. Let $f(x) = 2x$.
   (a) Graph several possible antiderivatives of f.
   (b) Graph the particular antiderivative F of f that satisfies $F(0) = -3$.

3. True or False? Explain your answer.
   (a) An antiderivative of a sum $f + g$ of functions f and g is a sum of an antiderivative of f and an antiderivative of g.
   (b) An antiderivative of a product fg of functions f and g is a product of an antiderivative of f and an antiderivative of g.

4. In general, what does it mean to say that a function solves an initial-value problem?

5. (a) Find an antiderivative of each function. Check your work by differentiating.

   (i) $f(x) = 8x^{15}$ (ii) $g(x) = 15e^{-5x}$ (iii) $h(x) = \cos 3x$

   (b) Evaluate the indefinite integral $\int (8x^{15} + 15e^{-5x} + \cos 3x)\, dx$.
   (c) Solve the initial-value problem $F'(x) = 8x^{15} + 15e^{-5x} + \cos 3x$, $F(0) = 2$.

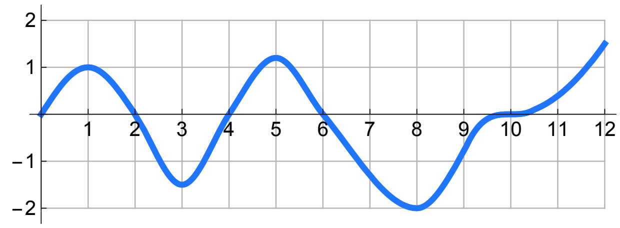
6. Show that the function $F(x) = \sin^{-1}\left(\frac{x}{2}\right) + \pi$ is a solution to the initial-value problem

$$F'(x) = \frac{1}{\sqrt{4 - x^2}}, \quad F(1) = \frac{7\pi}{6}.$$

7. Solve the initial-value problem $F'(x) = 10 \sin 5x + 3x^2$, $F(\pi) = 2$.

8. True or False: If $f'(x) = g'(x)$, then $f(x) = g(x)$.

9. The graph of $y = f(x)$ for a function f is shown on the right. Let F be a function defined on $[0, 9]$ that satisfies $F'(x) = f(x)$. In other words, F is an antiderivative of f.



   (a) On which intervals is F increasing? On which intervals is F decreasing?

   (b) On which intervals is the graph of F concave up? concave down?

   (c) Where does F have local maxima? local minima?

   (d) Where does the graph of F have inflection points?

   (e) Sketch the graph of the antiderivative F that satisfies $F(0) = 2$.

10. Find all antiderivatives of each function. Check your work by differentiating.

   (a) $f(x) = \dfrac{5}{\cos^2 x}$

   (b) $g(x) = \dfrac{1}{3x}$

   (c) $f(t) = \dfrac{t^2 - 2t + 5}{\sqrt{t}}$

   (d) $g(s) = 2^s$

Challenge Problems

1. If $f''(x) = 2e^x + 3\sin x$, $f'(0) = 0$, and $f(\pi) = 0$, calculate $f(x)$.

2. Suppose you are told that the acceleration function of an object is a continuous function $a(t)$. You are also given that the velocity $v(t)$ satisfies $v(0) = 1$.

   True or False: You can find the position of the object at any time t.

3. (a) Suppose $f'(x) = 6x^2$ and $f(1) = 1$. Explain why $f(-1) = -3$.

   (b) Suppose $f'(x) = x^{-2}$ and $f(1) = 1$. Is $f(-1) = 3$? Explain your answer.