Contents

2 Limits and Continuity

2.6 Continuity

1. Let f be a function defined on the interval $[a, b]$.
   (a) Suppose f is continuous on $[a, b]$. Also, suppose $f(a) > 0$ and $f(b) < 0$. What must be true about the function? What theorem justifies your answer?
   (b) Is your answer to part (a) true if f is not continuous on $[a, b]$? Explain or give a counterexample.
   (c) Is your answer to part (a) true if we assume that $f(a) < 0$ and $f(b) > 0$? Explain or give a counterexample.
2. Use your observations from Problem 1 to prove that the equation $x^4 + x - 3 = 0$ has a solution in the interval $(1, 2)$.
3. A mountain guide leaves the base camp at 8 a.m. and leads a group of hikers to the summit, where they arrive at 8 p.m. The group cooks dinner, gets some sleep, and then leaves the summit at 8 a.m. the next day. They return along the same path and arrive at the base camp at 8 p.m. Is there some point along the path that the guide crosses at exactly the same time on both days? Explain.
4. True or False? At some time since you were born your height in inches was exactly equal to your weight in pounds. Explain.

3 Derivatives

3.1 Introduction to the Derivative

1. Explain the difference between the two expressions

$$\frac{f(b) - f(a)}{b - a} \qquad \text{and} \qquad \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

2. Consider the function $f(x) = x^2 - 4x$.
   (a) Sketch the graph of f.
   (b) Without using calculus, sketch the line tangent to the graph of f at $x = 2$.
   (c) What is the slope of this tangent line?
   (d) Use the **definition** of the derivative to calculate $f'(x)$.
   (e) Evaluate $f'(2)$ using your answer from part (d).
   (f) How do your answers to parts (c) and (e) compare?

3. Suppose $y = 5x + 1$ is the line tangent to the graph of $y = f(x)$ for some function f at the point where $x = 2$. Calculate $f(2)$ and $f'(2)$.

4. Let f be the function $f(x) = x^3$.
   (a) Use the **definition** of the derivative to compute $f'(x)$.
   (b) Using your answer from part (a), evaluate the limit $\lim_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$.

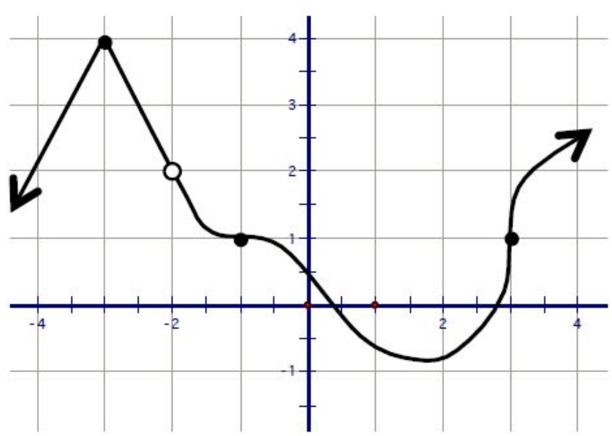
5. Let $g(x) = 1/(2x)$.
   (a) Use the **definition** of the derivative to compute $g'(x)$.
   (b) Use your answer from part (a) to find the slope of the line tangent to the graph of g at $x = 1$.
   (c) Write an equation for this tangent line.
   (d) Use your graphing calculator to plot the graph of $y = 1/(2x)$ and your answer from part (c).

6. Each of the expressions below represents the derivative of some function f at a number a. Determine such an f and a. Do NOT evaluate the limit. (Hint: Refer to the definition of the derivative or to Problem 1.)

   (a) $\lim_{h \to 0} \dfrac{\sqrt[5]{32+h} - 2}{h}$ (b) $\lim_{x \to 3} \dfrac{2^x - 8}{x - 3}$

3.2 Working with Derivatives

1. The graph of a function h is shown to the right. Use this graph to determine if h is differentiable at each of the indicated points. Justify your responses.



   (a) $x = -3$
   (b) $x = -2$
   (c) $x = -1$
   (d) $x = 3$

2. Sketch the graph of a function f that satisfies **all** of the following conditions:

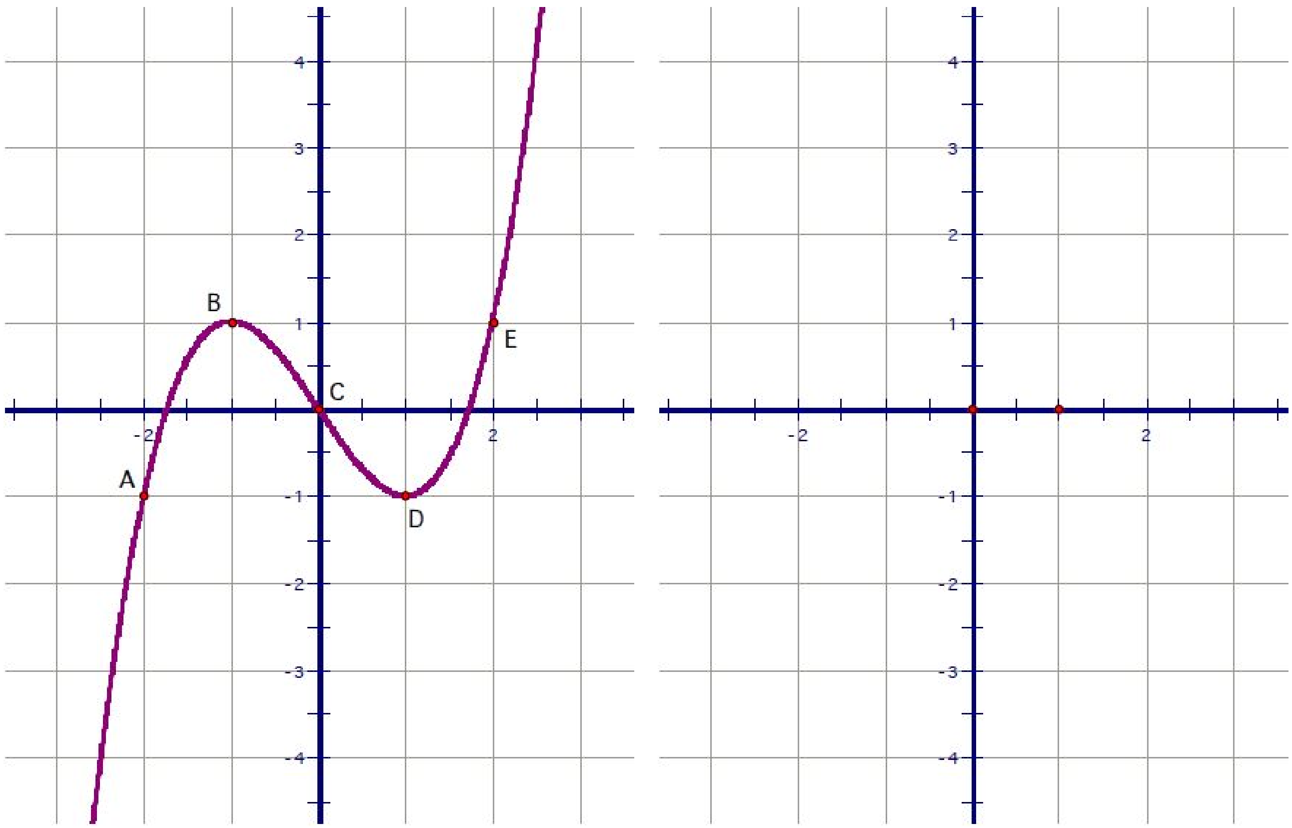
   (i) $f(0) = 0$ (ii) $f'(0) = -1$ (iii) $f'(2) = 0$ (iv) $f(3) = -2$

3. The graph of a function f is shown on the left below.

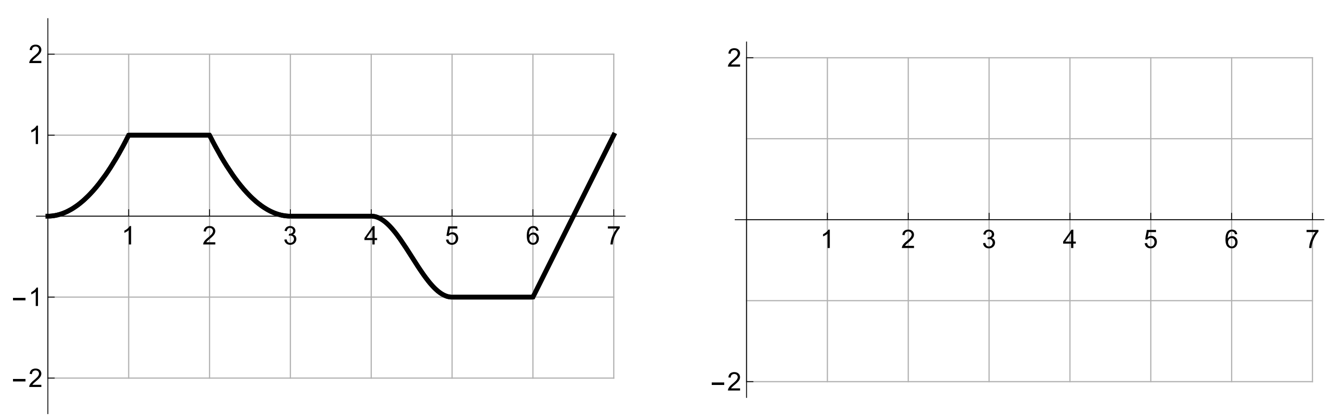
   (a) On the same figure, sketch the lines tangent to the graph at the points A, B, C, D, and E.

   (b) Approximate the slope of each of the tangent lines you sketched in part (a).

   (c) Make a rough sketch of the graph of the derivative f' on the axes provided on the right.



4. Given the graph of the function g shown on the left below, sketch the graph of the derivative of g on the axes provided on the right. (Use open dots to indicate the places where g is not differentiable.)



5. Consider the functions f whose derivative satisfies the conditions

$$f'(x) = \begin{cases} 1 & \text{if } x < 2, \\ 0 & \text{if } 2 < x < 4, \text{ and} \\ -1 & \text{if } x > 4. \end{cases}$$

   (a) Sketch the graph of one such function f that is continuous on the interval $[0, 6]$.

   (b) Are there any other such functions that are continuous on the interval $[0, 6]$? Explain your answer.

   (c) Sketch the graph of another such function f that is not continuous on the interval $[0, 6]$.

3.3 Rules of Differentiation

1. Let $f(x) = 1 + \sqrt{4x}$.

   (a) Use the definition of the derivative to calculate $f'(x)$.

   (b) Use the power rule to calculate $f'(x)$.

2. Calculate the derivative of $g(x) = (x^2 - 2)(3x - 1)$ by first simplifying the expression for $g(x)$ algebraically.

3. Calculate the derivative of $f(x) = \dfrac{3x^2 - 5x}{x^2}$ by first simplifying the expression for $f(x)$ algebraically.

4. Let $f(x) = \begin{cases} x^2 - 2x, & \text{if } x \le 3; \text{ and} \\ ax - b, & \text{if } x > 3. \end{cases}$

   Find the values of a and b such that f is differentiable at $x = 3$.

5. True or False? Explain your answer.

   (a) $\dfrac{d}{dx}(x^5) = 5x^4$ (b) $\dfrac{d}{dx}(e^5) = 5e^4$ (c) $\dfrac{d}{dx}(e^{5x}) = 5e^{4x}$

A. Challenge Problems

1. Determine an equation for a function with a vertical asymptote at $x = -2$, a removable discontinuity at $x = 1$, and a horizontal asymptote at $y = 0$.

2. Determine an equation for a function with a vertical asymptote at $x = 1$, a removable discontinuity at $x = -2$, and a horizontal asymptote at $y = 5$.

3. State the intervals of continuity for the function $f(x) = \sqrt{\dfrac{x}{x-2}}$.

4. Identify and classify the discontinuities of the functions

$$f(x) = \frac{|3x - 6|}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 + 4x - 12}{x^2 - 2x}.$$

5. Does $f(x) = x\sin(1/x)$ or $g(x) = \sin(1/x)$ have a removable discontinuity at $x = 0$? Justify your answer.

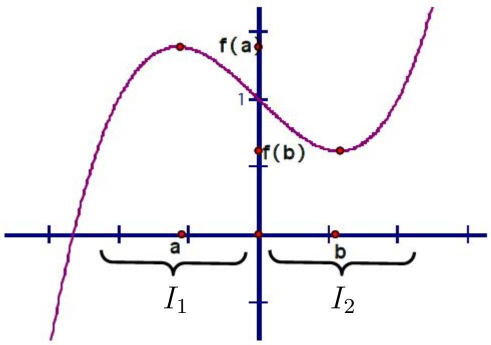
6. Consider the equation from Problem 2 of Section 2.6. Use a *bisection method* to approximate the value of the root that lies between $x = 1$ and $x = 2$ to the nearest hundredth. (That is, cut the interval $(1, 2)$ in half and determine if the solution lies in $(1, 1.5)$ or $(1.5, 2)$, then repeat this process.)

7. True or False? Explain. On the equator there are two diametrically opposed places that are exactly the same temperature at the same time.

8. Suppose I is an interval on which a function f is defined and c is a number in I. If $f(c) \geq f(x)$ for all x in I, then we say that $f(c)$ is a local maximum value of f. Similarly, if $f(c) \leq f(x)$ for all x in I, then we say that $f(c)$ is a local minimum value of f. This definition is illustrated here:

   (i) $f(a)$ is a local maximum value because $f(a)$ is greater than any other output on I_1.

   (ii) $f(b)$ is a local minimum value because $f(b)$ is less than any other output on I_2.



   (a) If f is differentiable on the entire real line, what must be true about the line tangent to the graph of f at a local maximum or minimum?

   (b) What does this mean about the derivative of f at a local maximum or minimum?

9. Suppose I is an interval on which the function f is defined. We say that f is *increasing* on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. Similarly, we say f is *decreasing* on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

   (a) If f is differentiable and increasing on an interval, what must be true of $f'(x)$ on this interval?

   (b) If f is differentiable and decreasing on an interval, what must be true of $f'(x)$ on this interval?

   (c) Suppose f is differentiable on the interval (a, c). If f is increasing on (a, b) and decreasing on (b, c), what must be true about the function at $x = b$?