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A. Challenge Problems **5**

Start by doing the following exercises:
§ 4.2: 1, 2, 3, 4, 6, 7, 8. § 4.3: 7, 8, 9, 10. § 4.4: 1, 2.

4 Applications of the Derivative

4.2 What Derivatives Tell Us

1. (a) Sketch the graph of a differentiable function f that is increasing and concave up for all x. What do we know about f' and f''?
   (b) Sketch the graph of a differentiable function f that is increasing and concave down for all x. What do we know about f' and f''?
   (c) Sketch the graph of a differentiable function f that is decreasing and concave up for all x. What do we know about f' and f''?
   (d) Sketch the graph of a differentiable function f that is decreasing and concave down for all x. What do we know about f' and f''?

2. Sketch the graph of a function f that satisfies all of the following criteria:
   (i) f is differentiable for all x.
   (ii) f is increasing and concave up on $(-\infty, -2)$.
   (iii) f is increasing and concave down on $(-2, 0)$.
   (iv) f is decreasing and concave down on (0,2).
   (v) f is decreasing and concave up on (2,4).
   (vi) f is decreasing and concave down $(4, \infty)$.

3. Provide an example of each of the following:
   (a) $f'(0) = 0$ but f' does not change sign at $x = 0$.
   (b) $f'(0) = 0$ but the second derivative test is inconclusive.
   (c) f has an inflection point at $x = 0$.
   (d) $f''(0) = 0$ but there is no inflection point at $x = 0$.

4. Determine whether the following statements are true or false. Provide an example or counterexample. (Rough graphs of $y = f(x)$ are acceptable.)
   (a) f' can remain constant while f'' changes.
   (b) f'' can remain constant while f' changes.
   (c) f' and f'' can both remain constant.
   (d) f' and f'' can both change.
   (e) f can only change concavity if $f'(x) \neq 0$.
   (f) f can have a critical point without changing concavity.

5. (a) Explain how to apply the First Derivative Test.
   (b) Explain how to apply the Second Derivative Test.

6. An apartment building has 100 rental units. The management company knows from experience that all apartments will be occupied if they charge \$800/month. A survey suggests that, on average, one additional apartment will remain vacant for every \$10 increase in rent. What rent should the management company charge to maximize revenue?

7. A boat leaves a dock at 2:00 P.M. and travels due south at a speed of 20 km/hr. Another boat has been heading due east at a speed of 15 km/hr and reaches the dock at 3:00 P.M. At what time were the two boats closest together?

8. For each of the following functions f, locate its critical values, inflection points, intervals where f is increasing/decreasing, intervals where its graph is concave up/down, local maxima and minima (if any), and absolute maxima and minima (if any).
   (a) $f(x) = x^3(x + 2)$ for all x
   (b) $f(x) = \sin x \cos x$ on the interval $[0, \pi]$
   (c) $f(x) = x^{2/3}(5/2 - x)$ for all x
   (d) $f(x) = \dfrac{x^2 - 4}{2x}$ on its domain

4.3 Graphing with Calculus

1. (a) How do you determine the x-intercepts of a function?
   (b) How do you determine the y-intercepts of a function?

2. (a) How do you locate the discontinuities of a function?
   (b) How do you classify discontinuities as *infinite* (vertical asymptotes) or *removable* (holes)?

3. How do you determine the domain of a function?

4. (a) How do you determine the end behavior of a function?
   (b) How do you know whether a function has a horizontal asymptote?
   (c) Describe the possible end behavior of a function that does not have a horizontal asymptote.

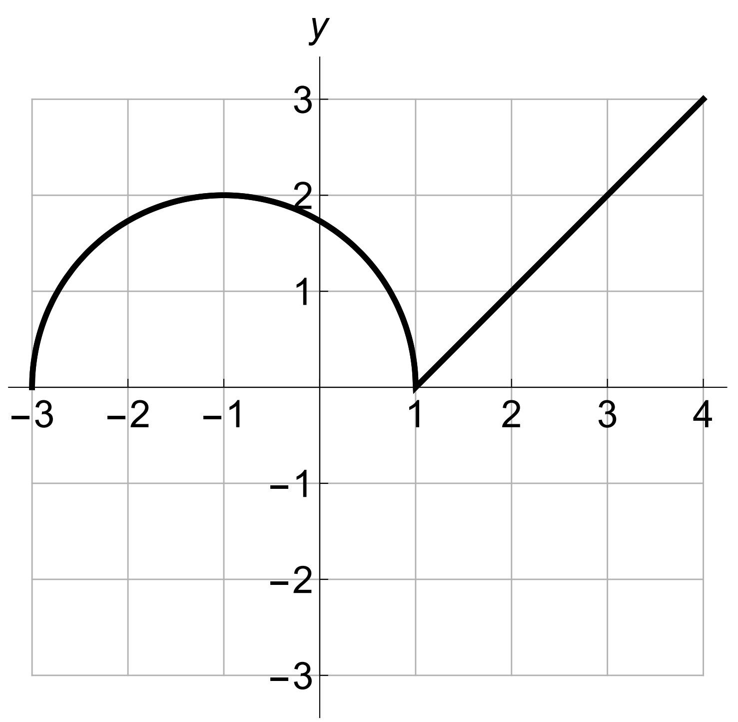
5. Make a complete graph of
$$f(x) = \frac{x^2 - 1}{x^2 - 4}.$$
Label all components of your graph including the x- and y-intercepts, the vertical and horizontal asymptotes, the local extrema, and the inflection points.

6. Identify the local and absolute extrema, the inflection points, and the x- and y-intercepts for the following three functions:

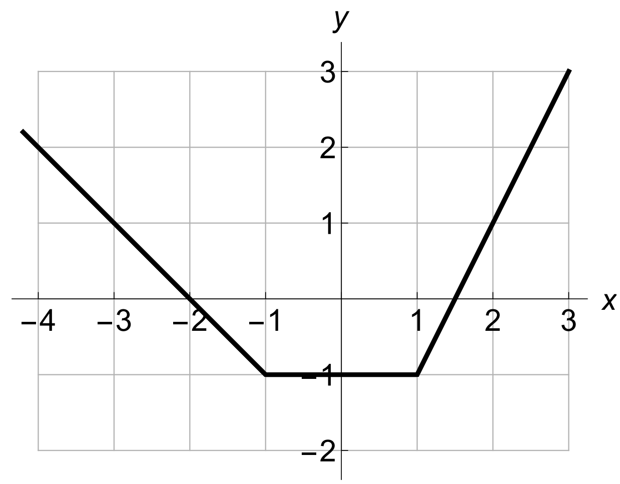
   (a) $f(x) = x + \cos x$ on $[0, 2\pi)$ (b) $g(x) = x^4 - 6x^2$ (c) $h(x) = e^{-(x-1)^2}$

7. Let $f(x) = x^2 e^{-x}$.
   (a) Find the critical points of f.
   (b) Determine the intervals on which f is increasing/decreasing.
   (c) Determine the intervals on which the graph of f is concave up/down.
   (d) Determine any local maximum/minimum values of f.
   (e) Sketch the graph of $y = f(x)$.

8. Consider the function graphed below. It is defined on the interval $[-3, 4]$. Sketch the graph of its derivative on the same set of axes. What is the derivative at $x = 1$? Why?



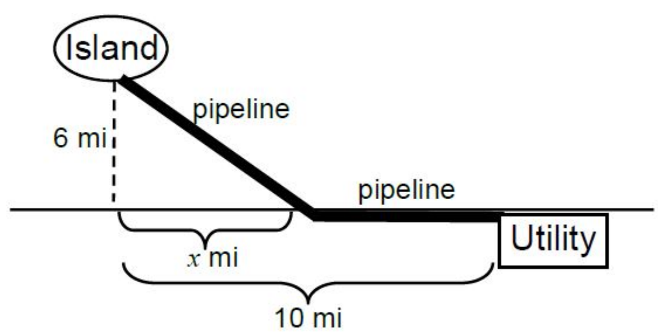
9. The graph of the *derivative*, $y = f'(x)$, of a function f is given below.
   (a) Sketch the graph of $y = f(x)$ on the same set of axes assuming that $f(0) = 0$. In other words, assume that the graph of f passes through the origin.
   (b) Why is it important to specify that f passes through the origin? How would your graph have been different if you were told that $f(0) = 1$?



10. Using all of the techniques discussed in this section, graph the function $f(x) = \dfrac{5}{e^x + 1}$.

4.4 Optimization Problems

1. A pipeline is to be constructed between an island and a utility company on the mainland. The island is 6 miles from shore and the utility company is 10 miles down the coast, as shown in the diagram below.



   (a) If it costs 1.25 times as much to construct the pipeline underwater as it does on land, find the value of x that minimizes the cost of construction.
   (b) If it costs 1.1 times as much to construct the pipeline underwater as it does on land, find the value of x that minimizes the cost of construction.
   (c) Explain how this example illustrates that it is necessary to check both the critical points and the endpoints when optimizing a function.

2. A factory is capable of producing 5,500 widgets per day.

   (a) If the cost of producing x widgets per day is given by

   $$C(x) = 25,000 + 0.04x + \frac{1,000,000}{x},$$

   find the number of widgets that will minimize daily production costs.

   (b) If the cost of producing x widgets per day is given by

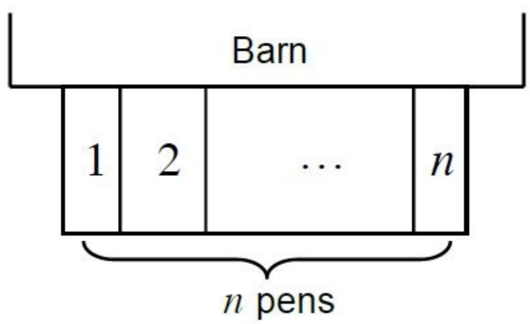
   $$C(x) = 25,000 + 0.04x + \frac{1,440,000}{x},$$

   find the number of widgets that will minimize daily production costs.

   (c) Explain how this example illustrates that it is necessary to consider the domain of the function you wish to optimize.

A. Challenge Problems

1. A farmer has L feet of fence available to build an enclosure consisting of n adjacent rectangular pens. The farmer will use the barn wall to form one side of the enclosure (as shown below). Show that the maximum area will always be achieved by allotting half of the available fence for the length and dividing the other half evenly among the widths.



2. Given an arbitrary line $y = mx + b$ and an arbitrary point (p, q) not on the line, find the point on the line that is closest to (p, q). Solve this question twice: first use only geometry and then use calculus.