Contents

Start by doing the following exercises:
§3.4: 3, 4, 5. §3.5: 1. §3.6: 2. §3.7: 1, 2, 3, 4, 5, 9, 10. §3.8: 3, 4, 5, 7.
§3.9: Precalculus 1 and 2; Calculus 3, 4, 5, and 6.

3 Differentiation

3.4 Product and Quotient Rules

1. Use the Product Rule to compute the derivative of $g(x) = (x^2 - 2)(3x - 1)$.

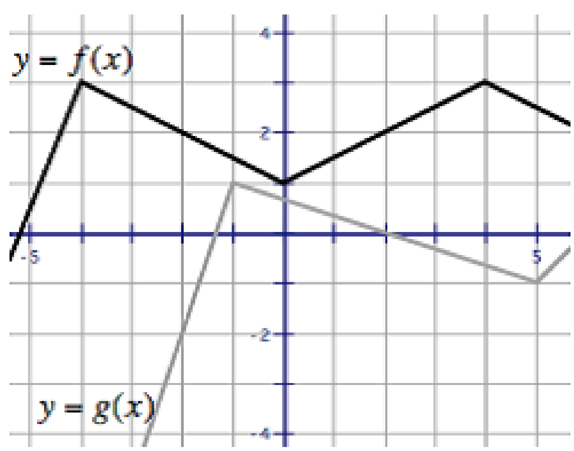
2. Use the Quotient Rule to find the derivative of $f(x) = \dfrac{3x^2 - 5x}{x^2}$.

3. Suppose $q(x) = f(x)/g(x)$. Use the table

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
10	4	-1	4	-5

to calculate $q'(10)$.

4. Suppose $p(x) = f(x)\,g(x)$. Use the graph to the right to calculate $p'(-2)$.



5. Calculate $\dfrac{d}{dx}\left[\left(\dfrac{7}{x} + e^{2x}\right)\left(4 + 3\sqrt[3]{x^2}\right)\right]$. (You do not need to simplify your answer).

6. The approximate number of bacteria in a lab experiment is given by $N(t)$ where t is time in hours.
   (a) What is the meaning of $N(5)$?
   (b) What is the meaning of $N'(5)$?
   (c) Given an unlimited amount of space and nutrients for the bacteria, what would you expect to be greater $N'(5)$ or $N'(10)$?
   (d) If the amount of space and nutrients for the bacteria were limited, would your answer to part (c) change?

7. Complete the following rules for differentiation:

   (a) $\frac{d}{dx}(c \cdot f(x))$ (b) $\frac{d}{dx}(f(x) + g(x))$ (c) $\frac{d}{dx}(f(x)g(x))$

   (d) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ (e) $\frac{d}{dx}(f(g(x)))$

Challenge Questions for Sections 3.3 and 3.4

1. Find a function f such that $f'(x) = 3x^2 + 6x + 5$. How many such functions f are there? Justify your answer.

2. Find a function $g(x)$ such that $g'(x) = \frac{1}{x^2} + \sqrt[3]{x}$ and $g(1) = 0$.

3. Use the Quotient Rule to prove the *Reciprocal Rule*: If g is differentiable and nonzero, then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2}$$

3.5 Derivatives of Trigonometric Functions

1. Derive an equation for the tangent line to the curve $y = e^x \cos x$ at the point where $x = 0$.

2. Compute $f^{(50)}(x)$ for (a) $f(x) = \sin x$ and (b) $f(x) = xe^{-x}$.

3. A 10-foot ladder rests against a vertical wall. Let θ be the angle formed by the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

4. Evaluate $\lim_{x \to 0} \frac{\sin(3x)}{x}$.

3.6 Derivatives as Rates of Change

Suppose the position s of an object moving horizontally after t seconds is given by a function f, that is, $s = f(t)$ where s is measured in feet, with $s > 0$ corresponding to positions to the right of the origin. For each of the two functions f below:

(a) Graph the position function.

(b) Find and graph the velocity function. When is the object stationary, moving to the right, and moving to the left?

(c) Determine the velocity and the acceleration of the object at $t = 1$.

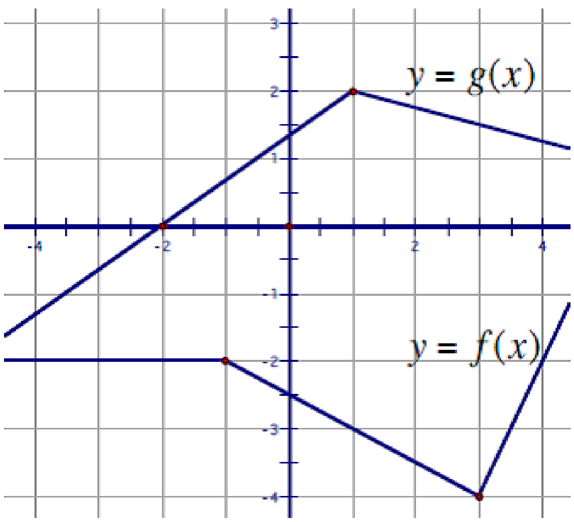
(d) Determine the acceleration of the object when its velocity is zero.

(e) On what intervals is the speed increasing?

1. $f(t) = t^2 - 4t, \quad 0 \le t \le 5$
2. $f(t) = 2t^2 - 9t + 12, \quad 0 \le t \le 3$

3.7 Chain Rule

1. If $h(x) = f(g(x))$, $g(-1) = 4$, $g'(-1) = 3$, $f(4) = 6$, $f'(4) = 5$, and $f'(-1) = 7$, compute $h'(-1)$.
2. Suppose $j(x) = g(f(x))$ and $k(x) = f(g(x))$. Use the graph to the right to evaluate $j'(4)$ and $k'(-4)$.



3. Evaluate (a) $\dfrac{d}{dx}\big(\tan(x^5)\big)$ and (b) $\dfrac{d}{dx}\big(\tan^5(x)\big)$.
4. Find all points on the graph of $f(x) = 2\sin x - \sin^2 x$ where the tangent line is horizontal.
5. Evaluate (a) $\dfrac{d}{dx}(1+e^{2x})(3x-6)^5$ and (b) $\dfrac{d}{dx}\left(\dfrac{e^{\sin x}}{\cos x}\right)$. (You do not need to simplify your answer.)
6. Evaluate $\dfrac{d}{dx}\sin^2(e^{3x})$.

7. Let $f(x) = (4x-5)^3$. Find $f'(x)$ in two ways:

   (a) Use the Chain Rule.

   (b) Expand $f(x)$ as a polynomial in standard form and then differentiate.

   Are your answers equivalent? Which method do you prefer?

8. Let $f(x) = \frac{1}{2}\sin(2x)$. Find $f'(x)$ in two ways:

   (a) Use the Chain Rule.

   (b) Use a trigonometric identity to re-write f as a product. Then use the Product Rule.

9. Let $f(x) = e^{3x^2-4x}$. Compute $f'(0)$ and write an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

10. Let $f(x) = \sqrt{\sin(x^3)}$.

    (a) Calculate $f'(x)$.

    (b) Explain why one might describe the example in part (a) as a "double chain rule."

11. If $y(x) = f(g(h(x)))$ where f, g, and h are arbitrary differentiable functions, derive a formula for the derivative $y'(x)$ in terms of f, g, and h and their derivatives (see Exercise 10).

Challenge Questions for Section 3.7

1. Evaluate the two limits: (a) $\displaystyle\lim_{x\to 2}\frac{\cos(x-2)-1}{x-2}$ and (b) $\displaystyle\lim_{x\to 2}\frac{\cos^2(x-2)-1}{x-2}$.

2. Let $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$

   (a) Show that f is continuous at $x = 0$

   (b) Use the definition of the derivative to show that $f'(0)$ does not exist.

   (c) Find $f'(x)$ for $x \neq 0$

   (d) Use a graphing calculator to graph f near $x = 0$.

3. Use the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to prove that

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

   (Hint: Start by multiplying the numerator and denominator by $(\cos x + 1)$.)

3.8 Implicit Differentiation

Note: In this section you do not need to simplify your answers unless you are instructed otherwise.

1. (a) Suppose $x^3 + (g(x))^2 = 9$. Compute $g'(x)$ in terms of x and $g(x)$.
   (b) Suppose $x^3 + y^2 = 9$. Compute dy/dx in terms of x and y.
   (c) Describe the similarities and differences between parts (a) and (b). How does this help explain the process of implicit differentiation?

2. Explain the difference between an *explicitly* defined function and an *implicitly* defined function.

3. (a) Find the coordinates of any points on the graph of $x^2 + y^2 = 25$ where $x = 3$.
   (b) Use implicit differentiation to find dy/dx for $x^2 + y^2 = 25$.
   (c) Find the slope of the line(s) tangent to the graph of $x^2 + y^2 = 25$ if $x = 3$.
   (d) Write the equation(s) of the line(s) tangent to the graph of $x^2 + y^2 = 25$ if $x = 3$.
   (e) Sketch the graphs of $x^2 + y^2 = 25$ and your answer(s) from part (d) on the same pair of axes.

4. Suppose $2xy + 3x^2 = 7y$ and we want to find dy/dx. As a first step we use implicit differentiation and find that $2y + 2x(dy/dx) + 6x = 7(dy/dx)$.

   (a) Now perform the second step and solve for dy/dx in terms of x and y.
   (b) Finally, solve for dy/dx in terms of x only.

5. Suppose $2xy+3x^2 = y^6+x$ and we want to find dy/dx. As a first step we use implicit differentiation and find that $2y + 2x(dy/dx) + 6x = 6y^5(dy/dx) + 1$.

   (a) Now perform the second step and solve for dy/dx in terms of x and y.
   (b) Why can't we solve for dy/dx in terms of x only?

6. Suppose $12\sqrt{x} - 2y = x/3$. Find dy/dx in two ways:

   (a) Use implicit differentiation.
   (b) Solve for y and then differentiate.

   Are your answers equivalent?

7. Consider the curve $2y^2 - x - y = 1$.

   (a) Find all points on this curve where the tangent line is vertical.
   (b) Does $2y^2 - x - y = 1$ ever have a horizontal tangent line? Explain.

Challenge Question for Section 3.8

1. Let $y = \sin^{-1} x$. Find dy/dx as follows:

   (a) Solve for x.
   (b) Use implicit differentiation to find dy/dx in terms of y.
   (c) Substitute $y = \sin^{-1} x$ into your result in part (b) to find dy/dx in terms of x.
   (d) Simplify the expression in the denominator. Hint: First try simplifying $\cos(\sin^{-1}(4/5))$.

3.9 Differentiation of Logarithms and Exponentials

Precalculus review questions on exponentials and logarithms

1. Solve the following equations:

   (a) $\log_{10} x = 3$ (b) $\log_b 125 = 3$ (c) $\ln x = -1$

   (d) $7^x = 21$ (e) $3^{3x-4} = 15$ (f) $5^{3x} = 29$

2. Convert the expressions to the indicated base:

   (a) 2^x converted to base e (b) $3^{\sin x}$ converted to base e

   (c) $\log_2(x^2+1)$ converted to base e (d) $\ln|x|$ converted to base 5

Calculus questions on exponentials and logarithms

1. (a) Complete the differentiation rule: $\frac{d}{dx}e^x =$
   (b) Suppose $b > 0$. Complete the differentiation rule: $\frac{d}{dx}b^x =$
   (c) Are these rules consistent? In other words, if you use the rule from part (b) to evaluate $\frac{d}{dx}e^x$, do you get the same answer as when you use the rule from part (a)?
2. (a) Complete the differentiation rule: $\frac{d}{dx}\ln x =$
   (b) Suppose $b > 0$. Complete the differentiation rule: $\frac{d}{dx}\log_b x =$
   (c) Are these rules consistent? In other words, if you use the rule from part (b) to evaluate $\frac{d}{dx}\ln x$, do you get the same answer as when you use the rule from part (a)?
3. Let $f(x) = \ln(5x^4 - 4x^3)$. Calculate $f'(x)$. On what interval is this calculation valid?
4. Calculate $g'(x)$ for the exponential function $g(x) = 5^{3x+6}$.
5. Calculate y'' for (a) $y = \dfrac{\ln(x)}{x^2}$ and (b) $y = x^2\ln(2x)$.
6. Let $h(x) = \log_3\left(\dfrac{x^2+1}{x+1}\right)$. Calculate $h'(x)$. On what interval is this calculation valid?
7. Let $f(x) = x^{3x}$. Use logarithmic differentiation to calculate $f'(x)$. (That is, start by taking the natural logarithm of both sides. Then differentiate with respect to x.)
8. Let $g(x) = \dfrac{e^{5x}\sqrt[3]{x^3+9}}{\cos^2(x)}$. Use logarithmic differentiation to calculate $g'(x)$.
9. In what cases should one choose to use logarithmic differentiation?
10. Calculate the derivative of $y = x^{\sqrt{x}} + 2^x \ln x$.

11. (a) Suppose $y = \ln(f(x))$. Write an expression for dy/dx in terms of $f(x)$. This expression is known as the *logarithmic derivative* of f. (Multiplying the logarithmic derivative by 100 gives us the percent rate of change of f.)

    (b) The function $p(t) = 300t + 6000$ can be used to model the gross domestic product (GDP) in billions of 1996 dollars where t is time in years since 1990. Find $p'(0)$ and $p'(5)$ and explain what these numbers tell us about GDP.

    (c) Find the percent rate of change of $p(t)$ at $t = 0$ and $t = 5$. Explain what these numbers tell us about GDP.

Challenge Question for Section 3.9

1. Use logarithmic differentiation and properties of logarithms to derive the Quotient Rule. Start with the expression

$$y = \frac{f(x)}{g(x)}.$$